11- Schumacher compression

16 września 2010
21:08

Recall Shannon coding theorem:

$$M^n \longrightarrow \boxed{\mathcal{E}} \xrightarrow{\quad X^k \quad} \boxed{D} \longrightarrow M'^n$$

message — n i.i.d $\{m, p(m)\}$; encoding; decoding

for faithful transmission we need $k = n H(M)$ bits. (We need to encode typical sequences

We want to formulate Quantum coding theorem – faithful compression of quantum states.

Let our „messages" be pure states $|\psi_m\rangle \in \mathcal{H}$ prepared with probability $p(m)$. Let us combine them into

$$\rho = \sum_m p(m)\, |\psi_m\rangle\langle\psi_m|.$$

So by analogy we can draw our problem:

$$\rho^{\otimes n} \longrightarrow \boxed{\mathcal{E}} \xrightarrow{\quad \sigma_k \quad} \boxed{D} \longrightarrow \rho'_n$$

encoding CP map; state of k qubits; we want ρ_n to represent faithfully transmitted states

Remark It is not enough to require that $F(\rho^{\otimes n}, \rho'_n) = 1$ since we know ρ we could just reprepare it at the output without using any channel. We need something more – we need to „transmit" quantum state – notice that similar issue arises in classical setting

$$\rho^{\otimes n} = \sum_{m_1 \ldots m_n} \underbrace{p(m_1)\cdot\ldots\cdot p(m_n)}_{p(m^n)} \underbrace{|\psi_{m_1}\rangle\langle\psi_{m_1}| \otimes \ldots |\psi_{m_n}\rangle\langle\psi_{m_n}|}_{|\Psi_{m^n}\rangle\langle\Psi_{m^n}|}$$

$$\sum_{m^n}$$

$$|\Psi_{m^n}\rangle\langle\Psi_{m^n}| \longrightarrow \rho'_{m^n} = D\left(\mathcal{E}\left(|\Psi_{m^n}\rangle\langle\Psi_{m^n}|\right)\right)$$

We want ρ'_{m^n} to be as close as possible to $|\Psi_{m^n}\rangle\langle\Psi_{m^n}|$

Average fidelity:

$$F = \sum_{m^n} p(m^n) \langle\Psi_{m^n}|\rho'_{m^n}|\Psi_{m^n}\rangle = \sum_{m^n} p(m^n)\, \mathrm{Tr}\left(|\Psi_{m^n}\rangle\langle\Psi_{m^n}|\rho'_{m^n}\right)$$

Reliable compression with rate R:

$$\forall_{\varepsilon}\ \exists_{n_0}\ \forall_{n \geq n_0}\ F \geq 1-\varepsilon \quad \text{where } k = n \cdot R$$

number of qubits used for encoding

11.1
Definition Von Neumann entropy

$$S(\rho) = -\mathrm{Tr}\,\rho \log \rho$$

Let $\rho = \sum_x p_x |x\rangle\langle x|$ be eigenvalue decomposition

$$S(\rho) = -\sum_x p_x \log p_x = H(X)$$

If we generate n i.i.d states according to $\{p(m), |\psi_m\rangle\}$ on average we have $\rho^{\otimes n}$. Eigenvalue decomposition

$$\rho^{\otimes n} = \sum_{x_1 \dots x_n} p_{x_1} \dots p_{x_n} |x_1\rangle\langle x_1| \otimes \dots |x_n\rangle\langle x_n|$$

Definition:

$|x_1\rangle \otimes \dots \otimes |x_n\rangle$ is ε-typical state iff
$x^n = \{x_1, \dots, x_n\}$ is ε-typical i.e.

$$\left| -\frac{1}{n} \log p(x^n) - S(\rho) \right| \leq \varepsilon$$

Definition

ε-typical subspace $T(n, \varepsilon)$, and projector $P(n, \varepsilon)$

$$P(n,\varepsilon) = \sum_{x^n\ \varepsilon\text{-typ}} |x_1\rangle\langle x_1| \otimes \dots \otimes |x_n\rangle\langle x_n|$$

Typ-subspace facts:

(i) $\forall_{\varepsilon} \forall_{\delta} \exists_{n_0} \forall_{n > n_0} \; Tr(P(n,\varepsilon)\rho^{\otimes n}) \geq 1 - \delta$

(ii) $\forall_{\varepsilon} \forall_{\delta} \exists_{n_0} \forall_{n > n_0} \; (1-\delta) 2^{n(S(\rho)-\varepsilon)} \leq Tr[P(n,\varepsilon)] \leq 2^{n(S(\rho)+\varepsilon)}$

(iii) $\forall_{R < S(\rho)} \forall_{\delta} \exists_{n_0} \forall_{n > n_0} \; Tr(Q(n)\rho^{\otimes n}) < \delta$
where $Q(n)$ projection on any 2^{nR} dimensional subspace of $H^{\otimes n}$

Proof:
(i) $Tr(P(n,\varepsilon)\rho^{\otimes n}) = \sum_{x_n \in \varepsilon\text{-typ}} p(x_n) \geq 1-\delta$ (by typical seq. properties) ■

(ii) $Tr[P(n,\varepsilon)] = \sum_{x_n \in \varepsilon\text{-typ}} = |A_\varepsilon^n|$ $\quad (1-\delta) 2^{n(S(\rho)-\varepsilon)} \leq |A_\varepsilon^n| \leq 2^{n(S(\rho)+\varepsilon)}$ ■

(iii) split the trace over typical and a-typical subspaces

$$Tr(Q(n)\rho^{\otimes n}) = Tr(Q(n)\rho^{\otimes n} P(n,\varepsilon)) + \underbrace{Tr[Q(n)\rho^{\otimes n}(I - P(n,\varepsilon))]}_{<\delta}$$

$[\rho^{\otimes n}, P(n,\varepsilon)] = 0$

because of (i)

$$Tr(Q(n)P(n,\varepsilon)\rho^{\otimes n} P(n,\varepsilon)) \leq 2^{nR}\ 2^{-n(S(\rho)-\varepsilon)}$$

upper bound on eigenvalues of $P(n,\varepsilon)\rho^{\otimes n}P(n,\varepsilon)$
(by typ. sequences $p(x^n) \leq 2^{-n(S(\rho)-\varepsilon)}$)

so if $R < S(\rho)$ this term goes to zero ■

11.2

Schumacher compression theorem:

Let $\rho = \sum_m p(m)|\psi_m\rangle\langle\psi_m|$ be i.i.d quantum source

(i) if $R > S(\rho)$ there exist a reliable compression at rate R

(ii) if $R < S(\rho)$ no reliable compression is possible

Proof:

(i) Let ε be such that $S(\rho) + \varepsilon \leq R$ by previous facts for sufficiently large n:

$$Tr(\rho^{\otimes n} P(n,\varepsilon)) > 1 - \delta$$

$$Tr(P(n,\varepsilon)) \leq 2^{nR} \quad (\dim T(n,\varepsilon) \leq 2^{nR})$$

Let H_c^n be a Hilbert space containing $T(n, \varepsilon)$

Let H_c^n be a Hilbert space containing $T(n,\varepsilon)$

- encoding:

projection on $\underbrace{P(n,\varepsilon)}$, $\underbrace{\mathbb{1} - P(n,\varepsilon)}$

error

we replace the state with some standard state

Let us define encoding CP map

$$\mathcal{E}: \mathcal{L}(H^{\otimes n}) \longrightarrow \mathcal{L}(H_c^n)$$

everything outside $\mathcal{H}_c^n$ goes to $|0\rangle\langle 0|$

$$\mathcal{E}(\varrho_n) = P(n,\varepsilon)\varrho_n P(n,\varepsilon) + \sum_i A_i \varrho_n A_i^{\dagger}$$

$A_i = |0\rangle\langle i|$ $\quad |i\rangle$ - basis in $\mathcal{H}_c^{n\perp}$

- decoding (identity on H_c^n)

$$\mathcal{D}(\sigma_k) = \sigma_k$$

$$\mathcal{D}(\mathcal{E}(\varrho^{\otimes n}))$$

What is the fidelity?

$$F = \sum_{m^n} p(m^n)\left[\operatorname{Tr}(|\psi_{m^n}\rangle\langle\psi_{m^n}| P(n,\varepsilon)|\psi_{m^n}\rangle\langle\psi_{m^n}|P(n,\varepsilon)) + \right.$$

$$\left. + \operatorname{Tr}(|\psi_{m^n}\rangle\langle\psi_{m^n}|\, |0\rangle\langle 0|) \cdot \operatorname{Tr}(|\psi_{m^n}\rangle\langle\psi_{m^n}| \cdot P^{\perp}(n,\varepsilon))\right] \geq$$

$$\geq \sum_{m^n} p(m^n) \underbrace{\operatorname{Tr}([|\psi_{m^n}\rangle\langle\psi_{m^n}| P(n,\varepsilon)\, |\psi_{m^n}\rangle\langle\psi_{m^n}| P(n,\varepsilon)]}_{A}$$

We know that $\operatorname{Tr}(\varrho^{\otimes n} P(n,\varepsilon)) \geq 1-\delta$

Let $|\psi_{m^n}\rangle = \lambda_{m^n}|\varphi_{m^n}\rangle + \mu_{m^n}|\xi_{m^n}\rangle$ $\qquad |\varphi\rangle \in T(n,\varepsilon) \quad |\xi\rangle \in T^{\perp}(n,\varepsilon)$

$$= \sum_{m^n} p(m^n)|\lambda_{m^n}|^4 = \sum_{m^n} p(m^n)(1-|\mu_{m^n}|^2)^2 \geq 1 - 2\sum_{m^n} p(m^n)|\mu_{m^n}|^2$$

Notice however that

$$\operatorname{Tr}(\varrho^{\otimes n} P(n,\varepsilon)) = \sum_{m^n} p(m^n)\operatorname{Tr}(|\psi_{m^n}\rangle\langle\psi_{m^n}| P(n,\varepsilon)) = 1 - \sum_{m^n} p(m^n)|\mu_{m^n}|^2$$

$$\geq 1-\delta \quad \Rightarrow \quad \sum_{m^n} p(m^n)|\mu_{m^n}|^2 \leq \delta \quad \Rightarrow$$

$$F \geq 1 - 2\delta$$

$$\begin{cases} F(\varrho^{\otimes n}, \mathcal{D}^n \circ \mathcal{C}^n) = |\operatorname{Tr}(\varrho^{\otimes n} P(n,\varepsilon))|^2 + \sum_i |\operatorname{Tr}(\varrho^{\otimes n} A_i)|^2 \geq \\ \geq |\operatorname{Tr}(\varrho^{\otimes n} P(n,\varepsilon))|^2 \geq |1-\delta|^2 \geq 1-2\delta \end{cases}$$

(ii)

Let $\mathcal{E}_i, \mathcal{D}_i$ be Kraus operators of coding, decoding operations

Let $Q(n)$ be projector on 2^{nR} coding subspace

$$F = \sum_{m^n} p(m^n)\operatorname{Tr}\left[|\psi_{m^n}\rangle\langle\psi_{m^n}| \sum_{ij} \mathcal{D}_i \mathcal{E}_j |\psi_{m^n}\rangle\langle\psi_{m^n}| \mathcal{E}_j^{\dagger}\mathcal{D}_i^{\dagger}\right]$$

$$|\psi_m\rangle = \lambda_m|\varphi_m\rangle + \mu_m|\xi_m\rangle$$

typ. subspace

$$|\psi_m\rangle\langle\psi_m| = \underbrace{|\lambda_m|^2|\varphi_m\rangle\langle\varphi_m|}_{P(n,\varepsilon)|\psi_m\rangle\langle\psi_m|P(n,\varepsilon)} + \underbrace{|\mu|^2|\xi_m\rangle\langle\xi_m| + \lambda_m\mu_m^{*}|\varphi_m\rangle\langle\xi_m| + h.c.}_{|\mu_m|\,\sigma_{m^n}}$$

$\|\sigma\| \leq 1$

$$F = \sum p(m^n)\operatorname{Tr}(P(n,\varepsilon)|\psi_{m^n}\rangle\langle\psi_{m^n}| P(n,\varepsilon) \sum_{ij}\mathcal{D}_i\mathcal{E}_j|\psi_{m^n}\rangle\langle\psi_{m^n}|\mathcal{E}_j^{\dagger}\mathcal{D}_i^{\dagger})$$

...

$$F = \sum_{m^n} p(m^n)\, Tr\Big(P(n,\varepsilon)\, |\psi_{m^n}\rangle\langle\psi_{m^n}|\, P(n,\varepsilon) \sum_{ij} D_i E_j |\psi_{m^n}\rangle\langle\psi_{m^n}| E_j^\dagger D_i^\dagger \Big)$$

$$+ \mu_{m^n}\, \underbrace{Tr\Big(\sigma_{m^n} \sum_{ij} D_i E_j |\psi_{m^n}\rangle\langle\psi_{m^n}| E_j^\dagger D_i^\dagger \Big)}_{\leq 1}$$

$$F \leq \underbrace{\sum_{ij}\sum_{m^n} p(m^n)\, \big| \langle\psi_{m^n}| P(n,\varepsilon) D_i E_j |\psi_{m^n}\rangle \big|^2}_{A} + \underbrace{\sum_{m^n} p(m^n)|\mu_{m^n}|}_{B}.$$

• B

We know that $Tr(\rho^{\otimes n} P(n,\varepsilon)) = 1 - \sum_{m^n} |\mu_{m^n}|^2 \geq 1-\delta$

$$\sum_{m^n} |\mu_{m^n}|^2 p_{m^n} \leq \delta$$

$$f = \sqrt{p(m^n)} \qquad g = \sqrt{p(m^n)}\, |\mu_{m^n}|$$

$$B \leq \sqrt{\sum_{m^n} p(m^n)} \cdot \sqrt{\sum_{m^n} p(m^n) |\mu_{m^n}|^2} = \sqrt{\delta} \xrightarrow{n \to \infty} 0$$

• Now consider A:

$$A \leq \sum_{ij}\sum_{m^n} p(m^n)\, \langle\psi_{m^n}| P(n,\varepsilon) D_i D_i^\dagger P(n,\varepsilon) |\psi_{m^n}\rangle \cdot \langle\psi_{m^n}| E_j^\dagger E_j |\psi_{m^n}\rangle$$

$$= Tr\Big(P(n,\varepsilon)\, \rho^{\otimes n}\, P(n,\varepsilon) \sum_i D_i D_i^\dagger \Big)$$

We know that $\sum_i D_i D_i^\dagger \geq 0$ and $Tr \sum_i D_i D_i^\dagger = Tr \sum_i D_i^\dagger D_i = 2^{nR}$

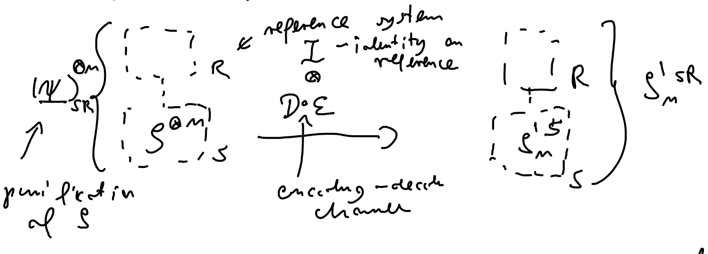
additionally eigenvalues of $P(n,\varepsilon)\rho^{\otimes n} P(n,\varepsilon)$ are $\leq 2^{-n(S(\rho)-\varepsilon)}$ so finally

$$\leq 2^{nR} \cdot 2^{-n(S(\rho)-\varepsilon)} \xrightarrow{n \to \infty} 0$$

∎

11.3 Another way of looking at quantum state transfer — preservation of entanglement

We think of ρ as reduced state of larger entangled state

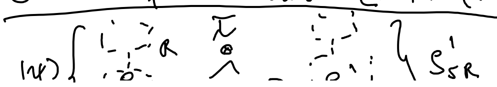


Our figure of merit is the so called Entanglement fidelity:

$$F(\rho^{\otimes n}, D \circ \mathcal{E}) = \langle\Psi|^{\otimes n}\, \rho'^{SR}_n\, |\Psi\rangle^{\otimes n}$$

We look how faithfully the purification is transmitted

<u>General facts about ent. fidelity</u>



$F(\rho, \Lambda) = \langle\psi| \rho'_{SR} |\psi\rangle$

$$\rho_S' = \sum_i K_i \rho K_i^\dagger$$

freedom of basis choice in R

$$\rho = \sum_k p_k |k\rangle\langle k| \qquad |\psi\rangle = \sum_k \sqrt{p_k}\, |k\rangle_S \otimes |e_k\rangle_R$$

$$\rho_{SR}' = \Lambda \circ \mathcal{I}\left(|\psi\rangle\langle\psi|\right) = \sum_i \left(\sum_k \sqrt{p_k}\, K_i |k\rangle \otimes |e_k\rangle\right)\left(\sum_{k'} \langle k'| K_i^\dagger \otimes \langle e_{k'}| \sqrt{p_{k'}}\right)$$

$$F(\rho, \Lambda) = \sum_i \left|\sum_k p_k \langle k|K_i|k\rangle\right|^2 = \sum_i |\mathrm{Tr}(\rho K_i)|^2$$

clearly does not depend on the purification

Returning to our setup we could reformulate Schumacher compression theorem as follows:

Schumacher compression theorem (entanglement preservation)

Let ρ be transmitted q-state. 1 Reliable transmission using compression to $k = n \cdot R$ qubits is such for which

$$\forall_{\varepsilon}\, \exists\, \forall_{n \, \text{maximum}} \; F(\rho^{\otimes n}, \mathcal{D} \circ \mathcal{E}) \geq 1 - \varepsilon$$

- If $R > S(\rho)$ reliable compression is possible
- If $R < S(\rho)$ no reliable compression exist.

Proof analogously with some minor modifications